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Remarks on entanglement measures and non-local state distinguishability

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Abstract

We investigate the properties of three entanglement measures that quantify the statistical distinguishability of a given state with the closest disentangled state that has the same reductions as the primary state. In particular, we concentrate on the relative entropy of entanglement with reversed entries. We show that this quantity is an entanglement monotone which is strongly additive, thereby demonstrating that monotonicity under local quantum operations and strong additivity are compatible in principle. In accordance with the presented statistical interpretation which is provided, this entanglement monotone, however, has the property that it diverges on pure states, with the consequence that it cannot distinguish the degree of entanglement of different pure states. We also prove that the relative entropy of entanglement with respect to the set of disentangled states that have identical reductions to the primary state is an entanglement monotone. We finally investigate the trace-norm measure and demonstrate that it is also a proper entanglement monotone.

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1. Introduction

Quantum entanglement arises as a joint consequence of the superposition principle and the tensor product structure of the quantum-mechanical state space of composite quantum systems. One of the main concerns of a theory of quantum entanglement is to find mathematical tools that are capable of appropriately quantifying the extent to which composite quantum systems are entangled. Entanglement measures are functionals that are constructed to serve that purpose [1–16]. Initially it was hoped for that a number of natural requirements reflecting the properties of quantum entanglement would be sufficient to establish a unique functional that quantifies entanglement in bi-partite quantum systems [4]. These requirements are the non-increase (monotonicity) of the functional under local operations and classical communication, the convexity of the functional (which amounts to stating that the loss of classical information does not increase entanglement) and the asymptotic continuity. Indeed, for pure quantum states these contraints (irrespective of convexity) essentially define a unique measure of entanglement. This uniqueness originates from the fact that pure-state entanglement can asymptotically be manipulated in a reversible manner [3] under local operations with classical communication (LOCC). However, for mixed states there is no such unique measure of entanglement, at least not under LOCC (see however[4] [17]). Instead, it depends very much on the physical task underlying the quantification procedure what degree of entanglement is associated with a given state. The distillable entanglement grasps the resource character of entanglement in mathematical form: it states how many maximally entangled two-qubit pairs can asymptotically be extracted from a supply of identically prepared quantum systems [3, 5]. The entanglement of formation [3, 6]—or rather its asymptotic version, the entanglement cost under LOCC [7, 19]—quantifies the number of maximally entangled two-qubit pairs that are needed in an asymptotic preparation procedure of a given state.

The relative entropy of entanglement [8–13] is an intermediate measure: it has an interpretation in terms of statistical distinguishability of a given state of the closest 'disentangled' state. This set of 'disentangled' states could be the set of separable states, or the set of states with a positive partial transpose (PPT states). The relative entropy of entanglement quantifies, roughly speaking, to what minimal degree a machine performing quantum measurements could tell the difference between a given state and any disentangled state [8].

It is not unthinkable that the optimal disentangled state may already be distinguishable from the primary state using selective local operations, rather than global ones. Yet, it would be interesting to see what measures of entanglement would arise if one considered only those disentangled states that cannot be distinguished locally from the primary state, specifically that both states have identical reductions with respect to both parts of the bi-partite quantum system. In this sense one asks for the degree to which the two states can be distinguished in a genuinely non-local manner.

It is the purpose of this paper to pursue this programme. We will discuss three different entanglement measures that are related to this distinguishability problem. Each of these entanglement measures is based on a different state space distance measure, namely on the relative entropy, the relative entropy with interchanged arguments and the trace-norm distance. The properties of these entanglement measures have not been studied so far. We will show that these three quantities are entanglement monotones, thereby qualifying them as proper measures of entanglement.

An interesting byproduct of this work is the result that the relative entropy of entanglement with interchanged arguments is strongly additive, which means that

$$E(\sigma \otimes \rho) = E(\sigma) + E(\rho) \tag{1}$$

for all states ρ and σ. Strong additivity implies weak additivity, i.e. $E(\rho^{\otimes n}) = nE(\rho)$ for all states ρ and all $n \in \mathbb{N}$. If one can interpret an entanglement measure as a kind of cost function, weak additivity can be interpreted as the impossibility to get a 'wholesale discount' on a state. Many measures of entanglement are known to be subadditive, such as the relative entropy of entanglement and the non-asymptotic entanglement of formation. Furthermore, all regularized asymptotic versions of entanglement measures are, by definition, weakly additive.

[4] Under PPT operations, that is, quantum operations preserving the positivity of the partial transpose, it is an open question whether there exists a unique measure of entanglement [17, 18]. In fact, there exist truly mixed states for which asymptotic state manipulation under PPT operations can be shown to be reversible [17], which points towards the possibility of having a unique measure of entanglement under PPT operations.

As no strongly additive measure of entanglement has been found so far, one might be led to doubt whether the requirements of (i) monotonicity, (ii) strong additivity and (iii) convexity are compatible at all. We will show, however, that the relative entropy of entanglement with interchanged arguments, and taken with respect to the set of disentangled states with the same reductions as the primary state, obeys each one of these three requirements, proving that there is no *a priori* incompatibility between them. It has to be noted, though, that this result is of a rather technical nature, as this measure of entanglement, while being physically meaningful, is not very practical: it yields infinity for any pure entangled state.

2. Notation and definitions

In this work we will consider bi-partite systems consisting of parts A and B, each of which is equipped with a finite-dimensional Hilbert space. The set of density operators of the joint system will be denoted as $\mathcal{S}(\mathcal{H})$. Let $\mathcal{D}(\mathcal{H})$ be either the set of separable states or the set of PPT states, which is the subset of $\mathcal{S}(\mathcal{H})$ which consists of the states σ for which the partial transpose σ^Γ is a positive operator. In the following, we will consider the proper subset $\mathcal{D}_\sigma(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})$ which consists of all those separable states (or PPT states) that are locally identical to σ,

$$\mathcal{D}_\sigma(\mathcal{H}) := \{\rho \in \mathcal{D}(\mathcal{H}) : \rho_A = \sigma_A, \rho_B = \sigma_B\}. \tag{2}$$

In this definition, subscripts A and B denote state reductions to the subsystems A and B, respectively. The quantities that will be considered in this paper are all distance measures with respect to this set:

$$E_A(\sigma) := \inf_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} S(\rho\|\sigma) \tag{3}$$

$$E_M(\sigma) := \inf_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} S(\sigma\|\rho) \tag{4}$$

$$E_T(\sigma) := \inf_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \|\rho - \sigma\|_1 \tag{5}$$

where

$$S(\rho\|\sigma) = \mathrm{tr}[\rho \log_2 \rho - \rho \log_2 \sigma] \tag{6}$$

is the relative entropy [20, 21], and $\|.\|_1$ stands for the trace norm [22].

The quantity E_M in equation (4) is the relative entropy of entanglement [8, 9] of a state σ with respect to the set $\mathcal{D}_\sigma(\mathcal{H})$. The original relative entropy of entanglement with respect to the set $\mathcal{D}(\mathcal{H})$ (meaning either separable or PPT states) is an entanglement measure that has been extensively studied in the literature [8, 9]. Initially formulated as a quantity for bi-partite finite dimensional systems, it has later been generalized to the asymptotic [10], the multi-partite [12] and the infinite-dimensional setting [13]. E_A in equation (3) is essentially the relative entropy with reversed entries, first mentioned in [8]. The particular property of this quantity is that it is strongly additive. The quantity E_T in equation (5) is a distance measure based on the trace norm. All quantities are related to the minimal degree to which a given bi-partite state σ can be distinguished from any state taken from $\mathcal{D}(\mathcal{H})$ that cannot be distinguished by purely local means with operations in A or B only. This statement will be made more precise in section 6.

The properties of E_A, E_M and E_T that will be investigated consist of the following well-known list of (non-asymptotic) properties of proper entanglement measures [3, 8, 15, 16, 4]:

(i) If $\sigma \in \mathcal{S}(\mathcal{H})$ is separable, then $E(\sigma) = 0$.
(ii) There exists a $\sigma \in \mathcal{S}(\mathcal{H})$ for which $E(\sigma) > 0$.
(iii) *Convexity*: Mixing of states does not increase entanglement: for all $\lambda \in [0, 1]$ and all $\sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$

$$E(\lambda\sigma_1 + (1-\lambda)\sigma_2) \leqslant \lambda E(\sigma_1) + (1-\lambda)E(\sigma_2). \tag{7}$$

(iv) *Monotonicity under local operations*: Entanglement cannot increase on average under local operations: if one performs a local operation in system A leading to states σ_i with respective probability p_i, $i = 1, \ldots, N$, then

$$E(\sigma) \geqslant \sum_{i=1}^{N} p_i E(\sigma_i). \tag{8}$$

(v) *Strong additivity*: Let $\mathcal{H}$ have the structure $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, with

$$\mathcal{H}^{(1)} = \mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)} \qquad \mathcal{H}^{(2)} = \mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}. \tag{9}$$

For all $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$ and $\sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$ then

$$E(\sigma^{(1)} \otimes \sigma^{(2)}) = E(\sigma^{(1)}) + E(\sigma^{(2)}). \tag{10}$$

For a thorough discussion of these properties, see [1, 4]. Functionals with the properties (i)–(iv) will as usual be denoted as entanglement montones.

3. Properties of E_A

The first statement that we will prove is the property of E_A to be an entanglement monotone in the above-mentioned sense, the second will be the strong additivity property.

Proposition 1. $E_A : \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ *with*

$$E_A(\sigma) := \inf_{\rho \in \mathcal{D}_\sigma(\mathcal{H})} S(\rho \| \sigma). \tag{11}$$

has the properties (i)–(iv), i.e., it is an entanglement monotone.

Proof. Properties (i) and (ii) are obvious from the definition, given that the relative entropy is not negative for all pairs of states. Let $\sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$, and let $\rho_1 \in \mathcal{D}_{\sigma_1}(\mathcal{H})$ and $\rho_2 \in \mathcal{D}_{\sigma_2}(\mathcal{H})$ be (not uniquely defined) states that are 'closest' to σ_1 and σ_2, respectively, in the sense that for $i = 1, 2$

$$E_A(\sigma_i) = S(\rho_i \| \sigma_i). \tag{12}$$

Such states always exist, due to the lower-semicontinuity of the relative entropy, and due to the fact that the sets $\mathcal{D}_{\sigma_1}(\mathcal{H})$ and $\mathcal{D}_{\sigma_2}(\mathcal{H})$ are compact. Then, for any $\lambda \in [0, 1]$,

$$\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{D}_{\lambda\sigma_1 + (1-\lambda)\sigma_2}(\mathcal{H}). \tag{13}$$

The convexity of E_A hence follows from the joint convexity of the relative entropy, and one obtains

$$\begin{aligned} \lambda E_A(\sigma_1) + (1-\lambda)E_A(\sigma_2) &= \lambda S(\rho_1 \| \sigma_1) + (1-\lambda)S(\rho_2 \| \sigma_2) \\ &\geqslant S(\lambda\rho_1 + (1-\lambda)\rho_2 \| \lambda\sigma_1 + (1-\lambda)\sigma_2). \end{aligned} \tag{14}$$

This is property (iii). The monotonicity of E_A under local operations can be shown as follows: as mixing can only reduce the degree of entanglement as measured in terms of E_A, it is sufficient to prove that equation (8) holds with

$$\sigma_i := (A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger / p_i \tag{15}$$

$$p_i := \mathrm{tr}[(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger] \tag{16}$$

where $A_i, i = 1, \ldots, N$, are operators satisfying $\sum_{i=1}^N A_i^\dagger A_i = \mathbb{1}$. Let $\rho \in \mathcal{D}_\sigma(\mathcal{H})$ be the state that satisfies $E_A(\sigma) = S(\rho\|\sigma)$. The state that is obtained after the measurement on ρ is given by

$$\rho_i := (A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger / \mathrm{tr}[(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger]. \tag{17}$$

As a consequence of $\rho \in \mathcal{D}(\mathcal{H})$ also

$$\rho_i \in \mathcal{D}_{\sigma_i}(\mathcal{H}) \tag{18}$$

holds for all $i = 1, \ldots, N$. The Kraus operators act in the Hilbert space of one party only and therefore,

$$\begin{aligned} p_i &= \mathrm{tr}[(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger] \\ &= \mathrm{tr}[(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger]. \end{aligned} \tag{19}$$

This is where the assumption that $\rho \in \mathcal{D}_\sigma(\mathcal{H})$ enters the proof. Then

$$\sum_{i=1}^N p_i S(\rho_i\|\sigma_i) = \sum_{i=1}^N \mathrm{tr}[(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger] S(\rho_i\|\sigma_i). \tag{20}$$

The right-hand side of equation (20) can now be bounded from above by $S(\rho\|\sigma)$, by virtue of an inequality of [21] (see also [8]), i.e.,

$$\sum_{i=1}^N \mathrm{tr}[(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger] S(\rho_i\|\sigma_i) \leqslant S(\rho\|\sigma). \tag{21}$$

Let $\omega_i \in \mathcal{D}_{\sigma_i}(\mathcal{H})$ be the state satisfying $E_A(\sigma_i) = S(\omega_i\|\sigma_i)$, then

$$E_A(\sigma) = S(\rho\|\sigma) \geqslant \sum_{i=1}^N p_i S(\omega_i\|\sigma_i) = \sum_{i=1}^N p_i E_A(\sigma_i). \tag{22}$$

This is property (iii), the monotonicity under local operations. □

Proposition 2. *E_A is strongly additive.*

Proof. Let $\mathcal{H}$ be a finite-dimensional Hilbert space with the above product structure $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, and let $\rho \in \mathcal{S}(\mathcal{H})$. From the conditional expectation property of the relative entropy [20] with respect to the partial trace projection it follows that

$$S(\rho\|\sigma^{(1)} \otimes \sigma^{(2)}) = S(\mathrm{tr}_2[\rho]\|\sigma^{(1)}) + S(\rho\| \mathrm{tr}_2[\rho] \otimes \sigma^{(2)})$$

for all $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)}), \sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$, such that

$$S(\rho\|\sigma^{(1)} \otimes \sigma^{(2)}) = S(\mathrm{tr}_2[\rho]\|\sigma^{(1)}) + S(\mathrm{tr}_2[\rho]\|\sigma^{(2)}) + S(\rho\| \mathrm{tr}_2[\rho] \otimes \mathrm{tr}_1[\rho]) \tag{23}$$

and hence

$$S(\rho\|\sigma^{(1)} \otimes \sigma^{(2)}) \geqslant S(\mathrm{tr}_2[\rho] \otimes \mathrm{tr}_1[\rho]\|\sigma^{(1)} \otimes \sigma^{(2)}). \tag{24}$$

Thus, it is always favourable to replace a given state by the product of its reductions with respect to 1 and 2.

Moreover, if $\rho \in \mathcal{D}_{\sigma^{(1)}\otimes\sigma^{(2)}}(\mathcal{H})$ for given $\sigma^{(1)} \in \mathcal{S}(\mathcal{H}^{(1)})$ and $\sigma^{(2)} \in \mathcal{S}(\mathcal{H}^{(2)})$, then also

$$\mathrm{tr}_2[\rho] \otimes \mathrm{tr}_1[\rho] \in \mathcal{D}_{\sigma^{(1)}\otimes\sigma^{(2)}}(\mathcal{H}). \tag{25}$$

This in turn implies that any 'closest' state $\rho \in \mathcal{D}_{\sigma^{(1)}\otimes\sigma^{(2)}}(\mathcal{H})$ that satisfies $E_A(\sigma^{(1)} \otimes \sigma^{(2)}) = S(\rho\|\sigma^{(1)} \otimes \sigma^{(2)})$ can be replaced by $\mathrm{tr}_2[\rho] \otimes \mathrm{tr}_1[\rho]$, which again satisfies

$$E_A(\sigma^{(1)} \otimes \sigma^{(2)}) = S(\mathrm{tr}_2[\rho] \otimes \mathrm{tr}_1[\rho]\|\sigma^{(1)} \otimes \sigma^{(2)}) = S(\mathrm{tr}_2[\rho]\|\sigma^{(1)}) + S(\mathrm{tr}_1[\rho]\|\sigma^{(2)}). \qquad (26)$$

Therefore,

$$E_A(\sigma^{(1)} \otimes \sigma^{(2)}) = E_A(\sigma^{(1)}) + E_A(\sigma^{(2)}) \qquad (27)$$

meaning that E_A is strongly additive. □

According to the statistical interpretation given in section 6, E_A has the property to be divergent for sequences of mixed states converging to pure states, and hence does not distinguish pure states in their degree of entanglement. Therefore, it is not a very practical measure of entanglement. However, as it is the only strongly additive entanglement monotone known to date, it appears fruitful to investigate the conditional expectation property of the relative entropy of entanglement further in order to try to construct strongly additive entanglement monotones that have the ability to discriminate between the degrees of entanglement of pure states.

4. Properties of E_M

In this section we will investigate the properties of the quantity E_M. First we will show that the relative entropy of entanglement E_M retains all properties of an entanglement monotone. In other words, the relative entropy of entanglement does not lose this property when additionally requiring the closest disentangled state to have the same reductions as the primary state. This observation implies a simplification when it comes to actually evaluating the relative entropy of entanglement, be it with analytical or with numerical means, because the dimension of the feasible set is smaller.

Proposition 3. *$E_M : \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ with*

$$E_M(\sigma) = \inf_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} S(\sigma\|\rho) \qquad (28)$$

is an entanglement monotone with properties (i)–(iv).

Proof. Properties (i), (i) and (iii) can be shown just as before. Again for states $\sigma, \sigma_1, \sigma_2 \in \mathcal{S}(\mathcal{H})$ and $\rho \in \mathcal{D}_\sigma(\mathcal{H})$ $\rho_1 \in \mathcal{D}_{\sigma_1}(\mathcal{H})$, $\rho_2 \in \mathcal{D}_{\sigma_2}(\mathcal{H})$ it follows that

$$A\rho A^\dagger/\,\mathrm{tr}[A\rho A^\dagger] \in \mathcal{D}_{A\sigma A^\dagger/\,\mathrm{tr}[A\sigma A^\dagger]}(\mathcal{H}) \qquad (29)$$

for all A, and

$$\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{D}_{\lambda\sigma_1+(1-\lambda)\sigma_2}(\mathcal{H}). \qquad (30)$$

With the notation of the proof of property (iv),

$$E_M(\sigma) = S(\sigma\|\rho) \geqslant \sum_{i=1}^N p_i S(\sigma_i\|\omega_i) = \sum_{i=1}^N p_i E_M(\sigma_i). \qquad (31)$$

□

Hence, the relative entropy of entanglement is still an entanglement monotone when one restricts the set of feasible PPT or separable states to those that are locally identical to a given state. At first it does not even seem obvious that E_M is even different from the original relative entropy of entanglement. In fact, all states σ considered in [8] satisfy

$$E_M(\sigma) = \inf_{\rho\in\mathcal{D}(\mathcal{H})} S(\sigma\|\rho). \qquad (32)$$

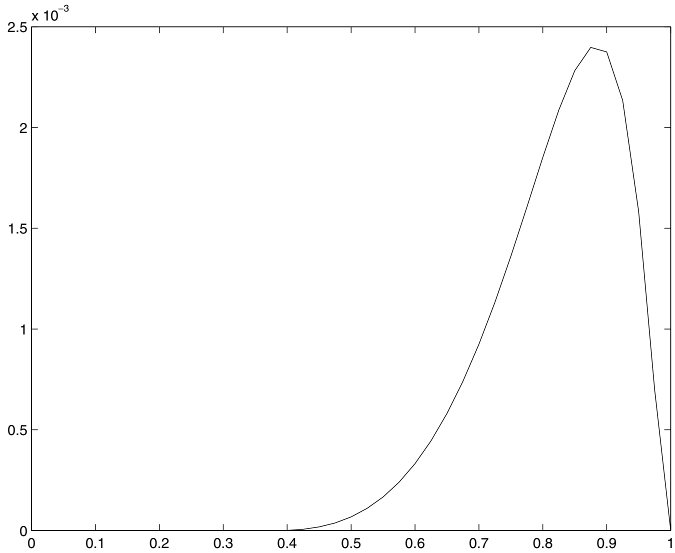

Figure 1. The difference $E_R(\rho_p) - E_M(\rho_p)$ for the state ρ_p as a function of p.

Also, for all UU and OO-symmetric states the two quantities are obviously the same. This version of the relative entropy of entanglement is strictly sub-additive, just as the relative entropy of entanglement with respect to the unrestricted sets of separable states or PPT states. However—on the basis of numerical studies—it turns out that the two quantities are not identical general, and that there exist states for which the two entanglement measures do not give the same value[5]. This means that the disentangled state that can be least distinguished from a given primary state may have the property that it can already be locally distinguished.

Example 4. We have numerically evaluated the difference $E_R(\rho_p) - E_M(\rho_p)$ between the (ordinary) relative entropy of entanglement E_R and the modified quantity E_M for states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ of the form

$$\rho_p := p|\psi\rangle\langle\psi| + (1-p)\mathbb{1}/4, \; p \in [0, 1], \tag{33}$$

where

$$|\psi\rangle := (|0,0\rangle + (1+i)|0,1\rangle + (1-i)|1,0\rangle)/5^{1/2}. \tag{34}$$

Figure 1 shows this difference $E_R(\rho_p) - E_M(\rho_p)$ as a function of $p \in [0, 1]$. The difference is in fact quite small, but significant, given the accuracy of the programme[6]. Numerical studies indicate that differences of this order of magnitude are typical for generic quantum states on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. Properties of E_T

We now turn to the third quantity E_T, the minimal distance of a state σ to the set $\mathcal{D}_\sigma(\mathcal{H})$ with respect to the trace-norm difference. We will show that also this quantity is a proper measure

[5] Reference [23] presents a set of equations that has to be satisfied for the closest state to have identical reductions as the primary state in the relative entropy of entanglement.

[6] The algorithm that has been used to numerically evaluate the two quantities will be discussed elsewhere.

of entanglement. Other physically interesting quantities of this type have been considered in the literature, in particular, the minimal Hilbert–Schmidt distance of a state to the set of PPT states [24–26]. For the latter quantity the resulting minimization problem can in fact be solved [24]. However, then the resulting quantity is unfortunately no proper entanglement measure [27].

Proposition 5. *$E_T : \mathcal{S}(\mathcal{H}) \longrightarrow \mathbb{R}^+$ with*

$$E_T(\sigma) = \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \|\sigma - \rho\|_1 \tag{35}$$

is an entanglement monotone with properties (i)–(iv).

Proof. Clearly, $E_T(\rho) = 0$ for a state $\rho \in \mathcal{D}(\mathcal{H})$. In order to show convexity one can proceed just as in the proofs of propositions 1 and 3: the convexity then follows from the triangle inequality for the trace norm. The remaining task is to show that it is monotone under local operations. Again,

$$p_i = \mathrm{tr}[(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger] = \mathrm{tr}[(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger] \tag{36}$$

for all $\rho \in \mathcal{D}_\sigma(\mathcal{H})$, and $(A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger/p_i \in \mathcal{D}_{\sigma_i}$. Hence,

$$\sum_{i=1}^N p_i E_T(\sigma_i) = \sum_{i=1}^N p_i \min_{\rho_i\in\mathcal{D}_{\sigma_i}(\mathcal{H})} \|(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger/p_i - \rho_i\|_1 \tag{37}$$

and since

$$\min_{\rho_i\in\mathcal{D}_{\sigma_i}(\mathcal{H})} \|(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger/p_i - \rho_i\|_1$$
$$\leqslant \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \frac{\|(A_i \otimes \mathbb{1})\sigma(A_i \otimes \mathbb{1})^\dagger - (A_i \otimes \mathbb{1})\rho(A_i \otimes \mathbb{1})^\dagger\|_1}{p_i} \tag{38}$$

we arrive at

$$\sum_{i=1}^N p_i E_T(\sigma_i) \leqslant \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \sum_{i=1}^N \|(A_i \otimes \mathbb{1})(\sigma - \rho)(A_i \otimes \mathbb{1})^\dagger\|_1. \tag{39}$$

Property (iv) then follows from lemma 6 (presented below), which yields

$$\begin{aligned}\sum_{i=1}^N p_i E_T(\sigma_i) &\leqslant \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \sum_{i=1}^N \|(A_i \otimes \mathbb{1})^\dagger(A_i \otimes \mathbb{1})|\sigma - \rho|\|_1 \\ &\leqslant \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \sum_{i=1}^N \mathrm{tr}[(A_i \otimes \mathbb{1})^\dagger(A_i \otimes \mathbb{1})|\sigma - \rho|] \\ &= \min_{\rho\in\mathcal{D}_\sigma(\mathcal{H})} \|\sigma - \rho\|_1 = E_T(\sigma).\end{aligned} \tag{40}$$

Hence, E_T is monotone under local operations. □

Lemma 6. *Let A, B be complex $n \times n$ matrices, and assume that $B = B^\dagger$. Then*

$$\|ABA^\dagger\|_1 \leqslant \|A^\dagger A|B|\|_1 \tag{41}$$

holds.

Proof. The trace norm $\|.\|_1$ is a unitarily invariant norm, and $ABA^\dagger$ is a normal matrix [22]. Hence

$$\|A(BA^\dagger)\|_1 \leqslant \|(BA^\dagger)A\|_1 \tag{42}$$

(see [22]), and therefore,

$$\|(BA^\dagger)A\|_1 = \mathrm{tr}[(A^\dagger AB^\dagger BA^\dagger A)^{1/2}] = \mathrm{tr}[(A^\dagger A|B|^2A^\dagger A)^{1/2}] = \|A^\dagger A|B|\|_1 \tag{43}$$

which gives rise to equation (41). □

Hence, E_T is a proper entanglement monotone, yet it does not exhibit an additivity property, and it is not asymptotically continuous on pure states. It should be noted that the weaker condition $E_T(\mathcal{E}(\sigma)) \leqslant E_T(\sigma)$ for all trace-preserving maps $\mathcal{E}$ corresponding to local operations with classical communication and all states σ follows immediately from the fact that the trace norm fulfils

$$\|\mathcal{E}(\sigma) - \mathcal{E}(\rho)\|_1 \leqslant \|\sigma - \rho\|_1 \tag{44}$$

for all trace-preserving completely positive maps $\mathcal{E}$ and all states σ, ρ. The Hilbert–Schmidt norm in turn does not have this property [27].

6. Distance measures and state distinguishability

In this section we will give an interpretation of the three quantities E_A, F_M and E_T in terms of hypothesis testing. The problem of distinguishing quantum-mechanical states can be formulated as testing two competing claims, see [28–30]. In this setup one considers a single dichotomic generalized measurement acting on a state that is known to be either ω or ξ, with equal *a priori* probabilities. The measurement is represented by two positive operators F and $\mathbb{1} - F$, with F satisfying $0 \leqslant F \leqslant \mathbb{1}$. On the basis of the outcome of the measurement one can then make the decision to accept either the hypothesis that the state ω has been prepared (the null-hypothesis), or the hypothesis that the state ξ has been prepared (the alternative hypothesis). The error probabilities of first and second kinds related to this decision are given by

$$\alpha(\omega, \xi; F) := \mathrm{tr}[\omega(\mathbb{1} - F)] \tag{45}$$

$$\beta(\omega, \xi; F) := \mathrm{tr}[\xi F]. \tag{46}$$

The trace-norm difference of the two states ω and ξ can be written in terms of these error probabilities as follows. According to the variational characterization of the trace norm,

$$\|\omega - \xi\|_1 = \max_{X, \|X\| \leqslant 1} \mathrm{tr}[(\omega - \xi)X] \tag{47}$$

where $\|.\|$ denotes the standard operator norm [22]. There is a one-to-one relation between the allowed X appearing here and the set of hypothesis tests: $F = (X + \mathbb{1})/2$. Hence, $\mathrm{tr}[(\omega - \xi)X] = 2\,\mathrm{tr}[(\omega - \xi)F]$ implying that the quantity E_T can be interpreted as

$$E_T(\sigma) = 2 \inf_{\rho \in \mathcal{D}_\sigma(\mathcal{H})} \max_{E} (1 - \alpha(\sigma, \rho; F) - \beta(\sigma, \rho; F)) \tag{48}$$

with F any test ($0 \leqslant F \leqslant \mathbb{1}$). Due to the restriction $\rho \in \mathcal{D}_\sigma(\mathcal{H})$, one compares the primary state σ only with those separable (PPT) ρ that have the same reductions as σ. Clearly, tests consisting of tensor products $F = F_A \otimes \mathbb{1}$ and $F = \mathbb{1} \otimes F_B$ cannot distinguish such states at all, as the outcomes will exhibit the same probability distributions for both states.

The quantum hypothesis tests related to E_T are restricted to a single measurement on a single bi-partite quantum system. The quantities F_M and E_A can in some sense be considered the asymptotic analogues of E_T. The connection between the relative entropy and the error probabilities in quantum hypothesis testing has been thoroughly discussed in [28–30]. In the asymptotic setting one considers sequences consisting of tuples of n identically prepared states, $\omega^{\otimes n}$ and $\xi^{\otimes n}$, and a sequence of tests $\{F_n\}_{n=0}^{\infty}$, where $0 \leqslant F_n \leqslant \mathbb{1}$ and F_n operates on an n-tuple. To every test in the sequence, one can again ascribe two error probabilities:

$$\alpha_n(\omega, \xi; F_n) := \mathrm{tr}[\omega^{\otimes n}(\mathbb{1} - F_n)] \tag{49}$$

$$\beta_n(\omega, \xi; F_n) := \mathrm{tr}[\xi^{\otimes n} F_n]. \tag{50}$$

For any $\varepsilon > 0$ define [29]

$$\beta_n^*(\omega, \xi; \varepsilon) := \min\{\beta_n(F_n) : 0 \leqslant F_n \leqslant \mathbb{1}, \alpha_n(\omega, \xi; F_n) < \varepsilon\}. \tag{51}$$

It has been shown [29] that for any $0 \leqslant \varepsilon < 1$

$$\lim_{n\to\infty} \frac{1}{n} \log \beta_n^*(\varepsilon) = -S(\omega\|\xi). \tag{52}$$

This means that if one requires that the error probability of first kind is no larger than ε, then the error probability of second kind goes to zero according to equation (52). Having this in mind, the quantity E_M can be interpreted as an asymptotic measure of distinguishing $\sigma \in \mathcal{D}(\mathcal{H})$ from the closest $\rho \in \mathcal{D}_\sigma(\mathcal{H})$ with the same reductions as σ. In turn, E_A is a similar quantity but with the roles of σ and ρ reversed. The asymmetry comes from the asymmetry of the roles of the error probabilities of first and second kinds.

Note that, within this interpretation, the divergence of E_A on pure states becomes plausible. If ξ is pure, choosing the sequence of tests $\{F_n\}_{n=0}^\infty$ with

$$F_n := \mathbb{1} - \xi^{\otimes n} \tag{53}$$

yields a β_n equal to zero for any n (this can only happen for pure ξ) and an α_n equal to $\mathrm{tr}[\omega\xi]^n$, which always becomes smaller than any chosen value of $\varepsilon > 0$ from some *finite* value of n onwards (that is, presuming $\omega \neq \xi$). Hence, for any choice of ε there is a finite value of n, say $n(\varepsilon)$, such that $\beta_n^*(\varepsilon) = 0$ for $n \geqslant n(\varepsilon)$. Asymptotical convergence of $\beta_n^*(\varepsilon)$ is therefore faster than exponential so that $\{\log \beta_n^*(\varepsilon)/n\}_{n=1}^\infty$ tends to minus infinity.

7. Summary and conclusion

In this paper we have investigated three variants of the relative entropy of entanglement, all three of which can be related to the problem of distinguishing a primary state from the closest disentangled or PPT state that has the same reductions as the primary state. This approach was motivated by the desire to flesh out the genuinely non-local distinguishability of a primary state from the closest disentangled state. The three functionals have been found to be legitimate measures of entanglement. Additionally, one functional has the property of being strongly additive, thereby showing that monotonicity, convexity and strong additivity are compatible in principle. This additivity essentially originates from the conditional expectation property of the relative entropy. In light of this observation it appears interesting to further study the implications of the conditional expectation property of the relative entropy on quantum information theory.

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References

[1] Horodecki M 2001 *Quant. Inf. Comp.* **1** 3
[2] Plenio M B and Vedral V 1998 *Contemp. Phys.* **39** 431
Werner R F 2001 Quantum information—an introduction to basic theoretical concepts and experiments *Springer Tracts in Modern Physics* (Heidelberg: Springer)

[3] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[4] Donald M J, Horodecki M and Rudolph O 2002 *J. Math. Phys.* **43** 4252
[5] Rains E M 1999 *Phys. Rev.* A **60** 173
[6] Wootters W K 2001 *Quant. Inf. Comp.* **1** 27
[7] Hayden P M, Horodecki M and Terhal B M 2001 *J. Phys. A: Math. Gen.* **34** 6891
[8] Vedral V and Plenio M B 1998 *Phys. Rev.* A **57** 1619
Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
[9] Vedral V, Plenio M B, Jacobs K A and Knight P L 1997 *Phys. Rev.* A **56** 4452
Rains E M 1999 *Phys. Rev.* A **60** 179
Donald M J and Horodecki M 1999 *Phys. Lett.* A **264** 257
Eisert J, Felbinger T, Papadopoulos P, Plenio M B and Wilkens M 2000 *Phys. Rev. Lett.* **84** 1611
Plenio M B, Virmani S and Papadopoulos P 2000 *J. Phys. A: Math. Gen.* **33** L193
Vollbrecht K G H and Werner R F 2001 *Phys. Rev.* A **64** 062307
[10] Audenaert K, Eisert J, Jane E, Virmani S, Plenio M B and de Moor B 2001 *Phys. Rev. Lett.* **87** 217902
[11] Audenaert K, De Moor B, Vollbrecht K G H and Werner R F 2002 *Phys. Rev.* A **66** 032310
[12] Plenio M B and Vedral V 2001 *J. Phys. A: Math. Gen.* **34** 6997
[13] Eisert J, Simon C and Plenio M B 2002 *J. Phys. A: Math. Gen.* **35** 3911
[14] Zyczkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 *Phys. Rev. A* **58** 883
Eisert J and Plenio M B 1999 *J. Mod. Opt.* **46** 145
Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314
Eisert J 2001 *PhD Thesis* University of Potsdam (see also [17] for an operational interpretation of the logarithmic negativity)
[15] Horodecki M, Horodecki P and Horodecki R 2000 *Phys. Rev. Lett.* **84** 2014
[16] Vidal G 2000 *J. Mod. Opt.* **47** 355
[17] Audenaert K, Plenio M B and Eisert J 2003 *Phys. Rev. Lett.* **90** (*Preprint* quant-ph/0207146) at press
[18] Audenaert K, Plenio M B and Eisert J 2003 *Phys. Rev. Lett.* **90** 027901
[19] Vidal G, Dür W and Cirac J I 2002 *Phys. Rev. Lett.* **89** 027901
[20] Ohya M and Petz D 1993 *Quantum Entropy and Its Use* (Heidelberg: Springer)
[21] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
[22] Bhatia R 1997 *Matrix Theory* (Heidelberg: Springer)
[23] Ishizaka S 2002 *J. Phys. A: Math. Gen.* **35** 8075
[24] Verstraete F, Dehaene J and De Moor B 2002 *J. Mod. Opt.* **49** 1277
[25] Bertlmann R A, Narnhofer H and Thirring W 2002 *Phys. Rev.* A **66** 032319
[26] Witte C and Trucks M 1999 *Phys. Lett.* A **257** 14
[27] Ozawa M 2000 *Phys. Lett.* A **268** 158
[28] Hiai F and Petz D 1991 *Commun. Math. Phys.* **143** 99
[29] Ogawa T and Hayashi M 2001 *Preprint* quant-ph/0110125
Ogawa T and Nagaoka H 2000 *IEEE Trans. IT* **46** 2428
[30] Fuchs C 1996 *PhD Thesis* University of New Mexico, Albuquerque